The Enveloping Semigroup and Stone-Čech compactification

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ABSTRACT: These are explanatory notes to myself on well-known subjects. This discusses: Nets, Ellis enveloping semigroup and the Stone-Čech compactification.

§A Nets

A ***directed-set*** I is a poset $(I, \preccurlyeq)$ such that

$\forall i,j \in I$, $\exists k \in I$ for which, simultaneously, $k \succcurlyeq i$ and $k \succcurlyeq j$.

A subset $S \subset I$ is ***eventual*** if

$$\exists k,\ \forall i \succcurlyeq k: \quad S \ni i\,.$$

In contrast, S is a ***frequent*** (or ***cofinal***) subset of I if

$$\forall k,\ \exists i \succcurlyeq k: \quad S \ni i\,.$$

Given directed-sets $(I, \preccurlyeq)$ and $(J, \preccurlyeq)$, the ***product directed-set*** $(I \times J, \leqslant)$ has partial-order

$$(i,j) \leqslant (i',j') \quad \text{IFF} \quad [i \preccurlyeq i' \ \&\ j \preccurlyeq j']\,.$$

We know that in a metric space,[♡1] the convergent sequences determine the topology. In a topological space X, the role of sequence is played by a more general object called a "net". A ***net***, in X, is a mapping from a directed-set I into X. Agree to write a net as $\langle x_i \rangle_{i \in I}$ or –suppressing I– as $\vec{x}$. To indicate that each x_i lies in a subset $A \subset X$, I may write $\vec{x} \subset A$.

Convergence

One writes

$$x_i \rightarrowtail y \quad \textit{or} \quad y = \lim_{i \in I} x_i \quad \textit{or} \quad y = \text{netlim}(\vec{x})$$

if: *For each open $U \ni y$, the set $\{i \mid x_i \in U\}$ is eventual.* One says that "the net $\vec{x}$ is eventually in each given neighborhood of y."

Similarly, y is an ***accumulation point*** of $\vec{x}$ if –for each nbhd $U \ni y$– the net is frequently in U.

Henceforth, let $\mathcal{N}[y]$ denote the directed-set of open nbhds of y. So in the directed-set $(\mathcal{N}[y], \preccurlyeq)$ the relation "$U \preccurlyeq V$" means $U \supset V$.

1: Lemma. *X is a topological space.*

a: *X is Hausdorff IFF net limits are unique.*

b: *For $A \subset X$: If y is an accumulation point of a net $\vec{a} \subset A$ then $y \in \overline{A}$. Conversely, if $y \in \overline{A}$ then there exists a net in A which converges to y.*

c: *A map $f: X \to \Omega$ is continuous IFF for each convergent net $\vec{x} = \langle x_i \rangle_i$ in X, its image $\langle f(x_i) \rangle_i$ is a convergent net in Ω. (We may write this image net as $f(\vec{x})$.)* ♢

Proof of (a). Suppose a net converges to both y and z. Given neighborhoods $U \in \mathcal{N}[y]$ and $V \in \mathcal{N}[z]$, the net is eventually in both U and V and so these nbhds are not disjoint.

Conversely, suppose that y and z do not have disjoint separate nbhds. Then for each pair $U \ni y$ and $V \ni z$ we can pick a point $x_{(U,V)}$ in $U \cap V$. The resulting net $\vec{x}$ is indexed by the product directed-set $\mathcal{N}[y] \times \mathcal{N}[z]$. And $\text{netlim}(\vec{x}) = y$ and $\text{netlim}(\vec{x}) = y$. ♦

Proof of (b). For the second assertion, suppose that $y \in \overline{A}$ and let $\mathcal{N} = \mathcal{N}[y]$. For each $V \in \mathcal{N}$ pick a point, call it a_V, in $A \cap V$. Then $\langle a_V \rangle_{V \in \mathcal{N}}$ forms a net converging to y. ♦

Proof of (c). To prove the ($\Rightarrow$) direction, suppose that net $\langle x_i \rangle_i$ converges to a point $z \in X$. For each nbhd V of $f(z)$, the set $f^{-1}(V)$ is a nbhd of z and so $\langle x_i \rangle_i$ is eventually in $f^{-1}(V)$. Hence $\langle f(x_i) \rangle_i$ is eventually in V. Thus $f(x_i) \underset{i}{\rightarrowtail} f(z)$.

For the ($\Leftarrow$) direction, suppose f not continuous. Then there is an open $V \subset \Omega$ and a point $z \in f^{-1}(V)$ such that *every* nbhd U of z "sticks out" of $f^{-1}(V)$ i.e, letting $\mathcal{N}$ denote $\mathcal{N}[z]$,

$$\forall\, U \in \mathcal{N}: \quad \textit{There exists a point } x_U \in U \smallsetminus f^{-1}(V).$$

[♡1] Indeed, this holds in each LCG (locally-countably generated) space.

By construction, then, $\langle x_U \rangle_{U \in \mathcal{N}}$ is a net in X converging to z. Yet the net $\langle f(x_U) \rangle_{U \in \mathcal{N}}$ cannot be converging to $f(z)$ since this net is included in the complement of V. ♦

Def: The analog of a subsequence. Consider two directed-sets $(S, \leqslant)$ and $(I, \preccurlyeq)$. A ***directed-set map*** (DSMap) is a map $\varphi: S \to I$ (not necessarily injective nor surjective) such that

†a: φ is order preserving: $s_1 \leqslant s_2 \implies \varphi(s_1) \preccurlyeq \varphi(s_2)$.

†b: The range, $\varphi(S)$, is frequent: *For all* $k \in I$ *there exists* $s \in S$ *with* $\varphi(s) \succcurlyeq k$.

A DSMap φ determines a subnet $\langle x_{\varphi(s)} \rangle_{s \in S}$. We might let a_s denote $x_{\varphi(s)}$ and write the subnet as $\vec{a} = \langle a_s \rangle_{s \in S}$.

Sometimes the map φ is implicit. For example, suppose that $\mathcal{N}$ is a directed-set. Then one subnet of $\langle x_i \rangle_{i \in I}$ is

$$\langle x_i \rangle_{(i,V) \in I \times \mathcal{N}} .$$

Here, our S is $I \times \mathcal{N}$, with product order, and φ is the "forgetful function" $(i, V) \mapsto i$. □

2: Theorem. *A point* $y \in X$ *is an accumulation point of net* $\langle x_i \rangle_{i \in I}$ *IFF there exists a subnet which converges to* y. ♢

Proof of ($\Rightarrow$). Let $\mathcal{N} = \mathcal{N}[y]$ and let $(I \times \mathcal{N}, \leqslant)$ be the product directed-set. Let S be the sub-poset consisting of those pairs (i, V) such that $x_i \in V$. Now $(S, \leqslant)$ is a directed-set: Suppose that (i, V) and (i', V') are in S. Pick $j \in I$ dominating i and i'. Since y is an accumulation point of $\langle x_j \rangle_{j \in I}$ there exists $k \geq j$ for which $x_k \in V \cap V'$. Thus $(k, V \cap V')$ is an element of S dominating both (i, V) and (i', V').

The net $\langle x_i \rangle_{(i,V) \in S}$ is a subnet of $\langle x_i \rangle_{i \in I}$ and it by definition converges to y. ♦

Proof of ($\Leftarrow$). Suppose $\varphi: S \to I$ is a DSMap such that

$$\lim_{s \in S} x_{\varphi(s)} = y .$$

Fixing a nbhd U of y, the set $E := \{s \mid x_{\varphi(s)} \in U\}$ is eventual in S. Thus its image, $\varphi(E)$, is frequent in I. ♦

3: Theorem. *X is compact IFF every net $\langle x_i \rangle_i$ has a convergent subnet.* ♢

Proof of ($\Rightarrow$). By the preceding theorem, it suffices to show that the net has an accumulation point. Suppose it does not. Then for each point $y \in X$ there is an open set $U \ni y$ for which the net fails to be frequently in U. So there exists $\gamma \in I$ with $x_i \notin U$ for every $i \succcurlyeq \gamma$.

Make explicit the dependence by writing U_y and γ_y. By compactness there is a finite set of points y, call it F, such that $\{U_y\}_{y \in F}$ covers X. But for each $i \in I$ which dominates *all* the $\{\gamma_y\}_{y \in F}$, we have the absurdity that x_i fails to be in $\bigcup_{y \in F} U_y$ —which equals X. ♦

Proof of ($\Leftarrow$). Assume X is non-compact and let $\mathcal{O}$ be a open cover with no finite subcover. Set

$$I := \{\mathcal{F} \subset \mathcal{O} \mid \mathcal{F} \text{ is finite}\} .$$

Since the union of two finite sets is finite, the pair $(I, \subset)$ is a directed-set. For each $\mathcal{F}$, the union $\bigcup \mathcal{F}$ is not all of the space and so we can choose a point

$$x_{\mathcal{F}} \in X \setminus \bigcup \mathcal{F} .$$

Could the net $\langle x_{\mathcal{F}} \rangle_{\mathcal{F} \in I}$ have an accumulation point $y \in X$? Fix some $V \in \mathcal{O}$ owning y. Then $\vec{x}$ fails to be frequently in V, since $x_{\mathcal{F}} \notin V$ for each $\mathcal{F} \ni V$ —that is, as soon as $\mathcal{F}$ is greater than the singleton $\{V\}$ in the partial order on I. So y is not an acc. point. ♦

§B Enveloping Semigroup

Consider a semigroup $\mathbb{H}$ and define, for each $g \in \mathbb{H}$, the right and left multiplications $R_g L_g : \mathbb{H} \to \mathbb{H}$ by $s \mapsto sg$ and $s \mapsto gs$. Evidently

$$R_g \circ R_h = R_{hg} \qquad L_g \circ L_h = L_{gh}$$

A ***half-topological semigroup*** $\mathbb{H}$ is a non-void compact Hausdorff topological space such that each R_g is continuous. Let $\mathbb{L}(\mathbb{H})$ denote the set of $g \in \mathbb{H}$ such that L_g is continuous. This $\mathbb{L}(\mathbb{H})$ is a sub-semigroup which is, generally, not compact. For each subset $H \subset \mathbb{H}$, let $\mathcal{Z}(H)$ (the *center* of H) denote the set of $\sigma \in H$ which commute with every member of H.

4: Lemma. *Suppose H is a subsemigroup of $\mathbb{L}(\mathbb{H})$ and let $\overline{H}$ denote the closure of H in $\mathbb{H}$.*

a: *$\overline{H}$ is a semigroup; hence it is an half-topological semigroup.*

b: $\mathcal{Z}(\overline{H}) \supset \mathcal{Z}(H)$. ♢

Proof of (a). For each pair $\beta,\zeta \in \overline{H}$, pick nets $\rho_i \rightarrowtail \beta$ and $\gamma_j \rightarrowtail \zeta$ in H. For each fixed i we have, since L_{ρ_i} is continuous, that $\rho_i\gamma_j \underset{j}{\rightarrowtail} \rho_i\zeta$. Hence $\overline{H} \supset \{\rho_i\zeta \mid i \in I\}$. But R_ζ is continuous and so

$$\beta\zeta \overset{\text{note}}{=\!=} \lim_{i \in I} \rho_i\zeta$$

is in $\overline{H}$. ♦

Proof of (b). Fix $\sigma \in \mathcal{Z}(H)$. Given $\beta \in \overline{H}$, we will show that $\sigma\beta = \beta\sigma$. Pick a net $H \ni \rho_i \rightarrowtail \beta$. Thus

$$\sigma\rho_i \rightarrowtail \sigma\beta\,, \qquad \text{since } \sigma \in \mathbb{L}(\mathbb{H});$$
$$\rho_i\sigma \rightarrowtail \beta\sigma\,, \qquad \text{since right multiplication is continuous.}$$

But $\sigma\rho_i$ equals $\sigma\rho_i$ by hypothesis. Thus the righthand sides are equal and so $\sigma \in \mathcal{Z}(\overline{H})$. ♦

An ***algebraic*** (left) ***ideal*** is a subset $I \subset \mathbb{H}$ satisfying $\mathbb{H}I \subset I$. An "ideal I" shall mean a non-empty algebraic ideal which is *compact*. Similarly, an ***algebraic*** sub-semigroup $H \subset \mathbb{H}$ satisfies $HH \subset H$ whereas a "semigroup H" will be, in addition, non-empty and compact. Thus a semigroup, now, is what we were previously calling a "half-topological semigroup". By Zorn's lemma, each semigroup $\mathbb{H}$ includes

a: a minimal ideal I.

b: a minimal sub-semigroup H.

5: Lemma. *Each minimal semigroup $\mathbb{H}$ is a singleton $\{\eta\}$; thus η is an idempotent element.* ♢

Remark. Each ideal I in a semigroup $\mathbb{H}$ is itself a semigroup. Thus each ideal contains an idempotent element. □

Notation: Agree to use "minimalness" as the property of a minimal ideal or sub-semigroup, in contrast to the "minimality" of a minimal set in X.

Proof. Fix an η in our minimal semigroup $\mathbb{H}$. Then $\mathbb{H}\eta \overset{\text{note}}{=\!=} R_\eta(\mathbb{H})$ is the continuous image of a compact set and is therefore itself compact. Since $\mathbb{H}\eta$ is a semigroup, and $\mathbb{H}\eta \subset \mathbb{H}$, minimalness implies that $\mathbb{H}\eta = \mathbb{H}$. Thus the set

$$H := \{\sigma \in \mathbb{H} \mid \sigma\eta = \eta\}$$

is non-empty. Now H is the inverse image of a closed set under a continuous map, since H is $R_\eta^{-1}(\{\eta\})$. Thus H is closed, hence compact. Evidently $HH \subset H$ and so the minimalness of $\mathbb{H}$ yields that $H = \mathbb{H}$. Hence $\eta \in H$ and is idempotent. ♦

6: Lemma. *Consider $T{:}X\circlearrowleft$, a continuous self-map of a compact Hausdorff topological space. Then*

a: *$(X^{\times X}, \circ)$ is a half-topological semigroup, where $\circ$ denotes composition.*

b: *For each $\alpha \in X^{\times X}$: The mapping $\eta \mapsto \alpha\eta$ is continuous IFF $\alpha{:}X \to X$ is continuous.* ♢

Proof of (a). $X^{\times X}$ is compact Hausdorff by Tychonoff's theorem and is a semigroup under composition. We need but check that right multiplication is continuous. So fixing $\alpha, \beta \in X^{\times X}$, we show that R_α is continuous at β.

In the product topology, "$\eta_i \rightarrowtail \beta$" is equivalent to "$\forall x \in X:\ \eta_i(x) \underset{i}{\rightarrowtail} \beta(x)$". This implies that

$$\forall x \in X:\ \eta_i(\alpha(x)) \underset{i}{\rightarrowtail} \beta(\alpha(x))\,.$$

So $\eta_i\alpha \underset{i}{\rightarrowtail} \beta\alpha$. Thus $\eta \mapsto \eta\alpha$ is continuous. ♦

Proof of (b). For the ($\Leftarrow$) direction, suppose α continuous. If $\eta_i \rightarrowtail \beta$ then for all x one has $\eta_i(x) \rightarrowtail \beta(x)$ and consequently $\alpha(\eta_i(x)) \underset{i}{\rightarrowtail} \alpha(\beta(x))$ by the continuity of α. Thus $\alpha\eta_i \rightarrowtail \alpha\beta$.

Conversely, for each convergent net $x_i \rightarrowtail y$ in X, there exists a convergent net $\eta_i \rightarrowtail \beta$ in $X^{\times X}$ with $\eta_i(y) = x_i$ and $\beta(y) = y$. (Just let β be the identity and let each η_i be the identity except at y.) If α is such that left multiplication is continuous, then $\alpha\eta_i$ must converge to $\alpha\beta$. *A fortiori* $\alpha\eta_i(y) \rightarrowtail \alpha\beta(y)$, i.e $\alpha(x_i) \underset{i}{\rightarrowtail} \alpha(y)$. This shows α to be continuous as a self-map of X. ♦

Definition. Given an algebraic semigroup H included in $\mathbb{L}(X^{\times X})$, its closure $\overline{H} \subset X^{\times X}$ is a semigroup by (4). The Ellis ***enveloping semigroup*** of T, written $E(T)$ or $E(X)$, is the closure of the powers $\{T^n \mid n \in \mathbb{Z}_+\}$ in $X^{\times X}$. So, "orbit" will mean "forward orbit". Agree to use $\mathcal{O}_T(x)$ or just $\mathcal{O}(x)$ to denote the (forward) orbit $\{T^n(x)\}_{n=1}^{\infty}$. Let $\overline{\mathcal{O}}(x)$ denote its closure. As an aside, by (4) the powers of T commute with every member of $E(T)$. □

Mirroring dynamical properties in the enveloping semigroup

Let $\mathbb{E}$ denote $E(X)$ and α, β, η denote elements of $\mathbb{E}$. For a subset $H \subset \mathbb{E}$, let $H(x)$ denote $\{\alpha(x) \mid \alpha \in H\}$. Thus $\mathbb{E}(x)$ equals $\overline{\mathcal{O}}(x)$.

Two points x, y are ***proximal***, written xPy, if $\overline{\mathcal{O}}_{T\times T}(x,y)$ intersects the diagonal. Thus xPy IFF there exists α such that $\alpha(x) = \alpha(y)$. For future use, recall that a ***distal point*** $x \in X$ is proximal, among points $y \in \overline{\mathcal{O}}(x)$, only to itself.

A point x is ***recurrent*** if for each neighborhood $U \ni x$ there exists a positive n with $T^n(x) \in U$. That is, there is a net such that $T^{n_i}(x) \underset{i}{\rightarrowtail} x$; hence, if $\beta(x) = x$ for some β.

Point x is ***almost-periodic*** if $\overline{\mathcal{O}(x)}$ is a minimal set, or equivalently: $y \in \overline{\mathcal{O}(x)} \implies x \in \overline{\mathcal{O}(y)}$. That is, if and only if $\forall\alpha, \exists\beta$ such that $\beta(\alpha(x)) = x$.

It is convenient to note the following.

‡a: For fixed $x, y \in X$: The (possibly empty) set $\{\alpha \in \mathbb{E} \mid \alpha(x) = \alpha(y)\}$ is a closed algebraic ideal.

‡b: With fixed $x \in X$ and sub-semigroup H: The (possibly empty) set $\{\beta \in H \mid \beta(x) = x\}$ is a closed algebraic semigroup.

7: Lemma. *If I is a minimal ideal of $\mathbb{E}$, then for each $\gamma \in I$:*

$$\forall x: \quad \gamma(x) \text{ is an almost-periodic point.}$$

Thus $I(x)$ is a minimal set. ♢

Proof. It suffices to show that for each $\alpha \in \mathbb{E}$ there is a $\beta \in \mathbb{E}$ such that $\beta\alpha\gamma = \gamma$. Indeed, we will show that β can be chosen from I.

Since $[I\alpha]\gamma \subset \mathbb{E}I \subset I$ and $I\alpha\gamma$ is compact, this $I\alpha\gamma$ is an ideal. Minimalness yields that $I\alpha\gamma = I$. Thus a β as stated exists.

The minimalness of I yields that $\mathbb{E}\gamma = I$. So

$$I(x) = \mathbb{E}\gamma(x) = \overline{\mathcal{O}(\gamma(x))},$$

a minimal set. ♦

8: Corollary (Auslander–Ellis theorem). *Every point $x \in X$ is proximal with an almost-periodic point in $\overline{\mathcal{O}(x)}$. Hence, each distal point is almost-periodic.* ♢

Proof. Pick $\eta \in I$, an idempotent in a minimal ideal. Then $\eta(x)$ is almost-periodic. And x is proximal with $\eta(x)$; just apply η to both. ♦

Characterization of recurrence by means of idempotents

Let $\mathbf{T}_n$ denote $\{T^k \mid k \geq n\}$ and $\overline{\mathbf{T}_n}$ its closure in $\mathbb{E}$; thus $\overline{\mathbf{T}_1}$ is another name for $\mathbb{E}$. Evidently

$$\begin{aligned} \mathbf{T}_1\overline{\mathbf{T}_n} &\subset \overline{\mathbf{T}_1\mathbf{T}_n} \quad \text{by left continuity} \\ &\subset \overline{\mathbf{T}_n}\,. \end{aligned}$$

Hence $\mathbb{E}\overline{\mathbf{T}_n} \subset \overline{\mathbf{T}_n}$, by right continuity; $\overline{\mathbf{T}_n}$ is an ideal.

Define $\Omega := \bigcap_{n=1}^{\infty} \overline{\mathbf{T}_n}$. Since Ω is the nested intersection of ideals, it is an ideal. Evidently $\Omega(x) := \{\alpha(x) \mid \alpha \in \Omega\}$ is the "Omega limit set" of x.

By its definition, $\mathbb{E} \smallsetminus \Omega \subset \mathbf{T}_1$. If a $T^{k_0} \in \mathbf{T}_1$ is idempotent, then $T^{k_0} = \lim_n T^{nk_0} \in \Omega$. Hence: *The semigroups $\mathbb{E}$ and Ω have exactly the same set of idempotents.*

9: Theorem. *In the following, η ranges over all idempotent elements in $\mathbb{E}$, hence in Ω.*

a: *$r \in X$ is recurrent IFF $\exists \eta$ with $\eta(r) = r$.*

b: *$a \in X$ is almost-periodic IFF $\exists/\forall$ minimal ideal(s) I in $\mathbb{E}$: $\exists \eta \in I$ with $\eta(a) = a$.*

c: *$d \in X$ is distal IFF $\forall \eta$: $\eta(d) = d$.* ◊

Proof of (a). If r recurrent then the set of β with $\beta(r) = r$ is non-void; hence a semigroup, by (‡b). So it owns an idempotent. ♦

Proof of (b), (⇒). The minimalness of I makes $I(a)$ a minimal set; hence the almost-periodicity of a insures that $a \in I(a)$. Thus $\{\beta \in I \mid \beta(a) = a\}$ is non-void and so is a sub-semigroup; which owns an idempotent.

The converse follows from (7). ♦

Proof of (c). Since $\eta(d) \in \overline{\mathcal{O}(d)}$, distality forces $\eta(d) = d$.

Conversely, suppose that d is proximal with a point $y := \beta(d)$. By (‡a), there is an idempotent α with $\alpha(y) = \alpha(d) = d$. The closed algebraic semigroup

$$\{\gamma \mid \gamma(d) = y \;\&\; \gamma(y) = y\}$$

is therefore non-empty, since $\beta\alpha$ is a member. Each member γ sends d to y. But there is an idempotent member which, by hypothesis, sends d to d. Hence $y = d$. ♦

The multiplier property

Say that point $x \in X$ is a ***recurrent multiplier*** if: For every system Z and recurrent point $z \in Z$, the pair-point $\langle x, z \rangle$ is recurrent (for the product system).

Similarly, x is a ***almost-periodic multiplier*** if it satisfies the above when "recurrent" is everywhere replaced by "almost-periodic".

10: Recurrent multiplier Theorem. *A point $d \in X$ is distal IFF it is a recurrent-multiplier.* ◊

Remark. The $\mathbb{E}$ below can either be viewed as the enveloping semigroup: of the product system $X \times Z$, *or* of the "plus 1" map on the Stone-Čech compactification of the natural numbers. This so-called "universal system" is developed below. □

Proof of (⇒). Given a recurrent point $z \in Z$, let η be an idempotent which fixes z. But $\eta(d) = d$ since d is distal. Hence $\eta(\langle d, r \rangle) = \langle d, r \rangle$. ♦

Proof of (⇐). We may assume that $X = \overline{\mathcal{O}}(d)$. Suppose, for the sake of contradiction, that d is proximal to some point $x \neq d$.

If d is almost-periodic, then x is. So we may assume that x is almost-periodic; since if d is not, then the Auslander–Ellis theorem assures us we can have found an x which is. So, taking an α for which $\alpha(x) = \alpha(d)$, we may assume that α fixes x. (Otherwise, just post-compose α with an member bringing $\alpha(x)$ to x.) The upshot is that $\alpha(\langle d, x \rangle) = \langle x, x \rangle$ in $X \times X$. That is, there exists a net $(m_i)_i$ such that

$$T^{m_i}(d) \underset{i}{\twoheadrightarrow} x \quad \text{and} \quad T^{m_i}(x) \underset{i}{\twoheadrightarrow} x.$$

IP set

For a collection $\mathcal{C}$ of numbers let $\text{FS}(\mathcal{C})$ denote the set of all finite sums $\sum_{n \in \mathcal{F}} n$, where $\mathcal{F}$ ranges over all finite subsets of $\mathcal{C}$.

Fix disjoint open sets $D \ni d$ and $V \ni x$. Inductively choose positive integers $n_1 < n_2 < \ldots$ so that, at stage K:

$P(K)$: *For each non-zero $N \in \text{FS}(\{n_k\}_1^K)$ we have that $T^N(d) \in V$ and $T^N(x) \in V$.*

Thus $\lim_i T^{N+m_i}(d) = T^N(\lim_i T^{m_i} d) = T^N(x) \in V$, and the same holds true for "d" replaced by "x". So we can pick a "sufficiently large" term $n_{K+1} \in \{m_i\}_i$ so that $(P(K+1))$ holds. And so that

11: $$n_{K+1} > n_1 + n_2 + \cdots + n_K\,.$$

Recurrent point

Let P denote the IP seq $\text{FS}((n_k)_{k=1}^{\infty})$. Define $Z := \{0,1\}^{\mathbb{N}}$ with the product topology and with the shift S

acting. Define a point r to be the indicator function $\mathbf{1}_P$. Condition (11) yields that $S^{n_k}(r) \to r$ as $k \to \infty$.

Since r is recurrent, the hypothesis on d says that the pair $\langle d, r\rangle$ must be recurrent. Hence there exists a positive p such that $T^p(d) \in D$ and $\text{Dist}(r, S^p r) < 1$. This latter condition, since $r = \mathbf{1}_P$, forces p to be in the IP-sequence P. n Consequently (P) implies that $T^p(d) \in V$. Contradiction. ♦

Factors and the enveloping semigroup

Suppose we have systems $(T\colon X)$ and $(S\colon Y)$ and a factor map $\psi{:}Y{\to}X$. (That is, ψ is a continuous surjection with $T\psi = \psi S$.) Let $\mathbb{E}$ denote $E(Y)$ and $\check{\mathbb{E}} := E(X)$. Given a $\beta \in \mathbb{E}$, write it as a net-limit $S^{m_i} \rightarrowtail \beta$. We wish to construct the righthand diagram below.

12:
$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ \psi\big\downarrow & & \psi\big\downarrow \\ X & \xrightarrow{T} & X \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{\beta} & Y \\ \psi\big\downarrow & & \psi\big\downarrow \\ X & \xrightarrow{\check{\beta}} & X \end{array}$$

Fix $y \in Y$. Since ψ is continuous,

$$\psi(\beta y) \;=\; \lim_i \psi(S^{m_i} y) \;=\; \lim_i T^{m_i}\psi(y)\,.$$

So, since ψ is surjective, $\check{\beta} := \lim_i T^{m_i}$ exists. And

13:
$$\psi\beta = \lim_i T^{m_i}\psi = \check{\beta}\psi,$$

where the last step follows by continuity right multiplication, R_ψ.

This $\check{\beta}$ is unique. For if net S^{n_j} also converges to β then $\beta' := \lim_j T^{n_j}$ exists. By (13) used twice,

$$\beta'\psi = \psi\beta = \check{\beta}\psi.$$

But ψ is onto. Thus $\beta' = \check{\beta}$.

14: Theorem. *Suppose T is a factor of S as in figure (12). Then there exists a unique continuous surjective semigroup-homomorphism $\Psi\colon \mathbb{E} \twoheadrightarrow \check{\mathbb{E}}\colon \beta \mapsto \check{\beta}$ such that $\check{\beta}\psi = \psi\beta$ and $\check{S} = T$.* ♢

Proof. Surjectivity of Ψ: Given a convergent net $\gamma := \lim T^{n_j}$, some subnet of $\{S^{n_j}\}_j$ converges; to a member of $\Psi^{-1}(\gamma)$.

To check that Ψ is a homomorphism fix $\alpha, \beta \in \mathbb{E}$. Then by (13) used three times,

$$[\beta\alpha]^\vee\,\psi \;=\; \psi\beta\alpha \;=\; \check{\beta}\psi\alpha \;=\; \check{\beta}\check{\alpha}\psi\,.$$

Since ψ is surjective, then, $[\beta\alpha]^\vee = \check{\beta}\check{\alpha}$.

Finally, to verify that Ψ is continuous, fix a convergent net $\alpha_i \rightarrowtail \beta$ in $\mathbb{E}$. The continuity of ψ followed by (13) yields

$$\lim_i \psi\alpha_i \;=\; \psi\beta \;=\; \check{\beta}\psi\,.$$

Hence $\lim_i \check{\alpha}_i\psi = \check{\beta}\psi$ exists. But ψ is surjective and so $\lim_i \check{\alpha}_i$ exists and equals $\check{\beta}$. ♦

The universal point-transitive system

(In this section all maps are continuous and all spaces are Hausdorff.) Let $\mathbb{N}\overset{\varphi}{\hookrightarrow}\widehat{\mathbb{N}}$ be the canonical embedding of the set of natural numbers into its Stone-Čech compactification. Each map $f{:}\mathbb{N}{\to}K$ into a compact space has a unique ***lift*** $\hat{f}{:}\widehat{\mathbb{N}}{\to}K$ such that $\hat{f}\circ\varphi = f$.

Each function $P{:}\mathbb{N}{\to}\mathbb{N}$ is necessarily continuous and so $p := \varphi\circ P$ is continuous and lifts to a map $\hat{p}\colon \widehat{\mathbb{N}} \to \widehat{\mathbb{N}}$ making this diagram commute:

15:
$$\begin{array}{ccc} \widehat{\mathbb{N}} & \xrightarrow{\hat{p}} & \widehat{\mathbb{N}} \\ \varphi\big\uparrow & & \varphi\big\uparrow \\ \mathbb{N} & \xrightarrow{P} & \mathbb{N} \end{array}$$

Our application will be when P is the "plus 1" map $n \mapsto n+1$, and $\hat{p}$ is its correspondent in the Stone-Čech compactification.

16: Universal-lift Theorem. *Suppose $x_0 \in X$ is a transitive point for system $(T\colon X)$. Then there exists a unique factor map $\hat{\psi}\colon \widehat{\mathbb{N}} \to X$ making T a factor of $\hat{p}$ with $\hat{\psi}(0) = x_0$.* ♢

Proof. The continuous map $\psi{:}\mathbb{N}{\to}X : n \mapsto T^n(x_0)$ intertwines P with T. Then its lift, $\hat{\psi}$, is the desired factor map. For the three commutativity relations (15), $T\psi = \psi P$ and $\hat{\psi}\varphi = \psi$ give that this diagram is commutative,

17:
$$\begin{array}{ccc} \widehat{\mathbb{N}} & \xrightarrow{\hat{p}} & \widehat{\mathbb{N}} \\ {\scriptstyle\hat{\psi}}\big\downarrow & & {\scriptstyle\hat{\psi}}\big\downarrow \\ X & \xrightarrow{T} & X \end{array}$$

when restricted to the image $\varphi(\mathbb{N})$ —which is a *dense* subset of $\widehat{\mathbb{N}}$. Hence the diagram commutes. ♦

18: Corollary. *The enveloping semigroup $E(\widehat{\mathbb{N}})$ of the shift on the Stone-Čech compactification of the natural numbers acts on all point-transitive systems.* ♢

Uniqueness

Consider the category of triples $(T\colon X, x_0)$ where x_0 is a transitive point for T. A ***morphism***,

$$(S\colon Y, y_0) \xrightarrow{\psi} (T\colon X, x_0)$$

in this category, is a factor map $(S\colon Y) \xrightarrow{\psi} (T\colon X)$ sending $y_0 \mapsto x_0$. Given two such triples, the morphism ψ is unique, since x_0 has dense orbit.

Suppose, in addition to ψ, we have a morphism in the other direction:

$$(S\colon Y, y_0) \xleftarrow{\xi} (T\colon X, x_0)\,.$$

Then $\xi \circ \psi$ is an automorphism of $(S\colon Y, y_0)$ which, by uniqueness, must be the identity map. Similarly, $\psi \circ \xi$ is the identity on X. Consequently ψ and ξ are isomorphisms of the two triples.

Applying this to two potential universal-lifts yields this unsurprising conclusion: *The universal-lift $(\hat{p}\colon \widehat{\mathbb{N}}, 0)$ is unique up to isomorphism.*

Realizing $E(\widehat{\mathbb{N}})$ using the full-shift

Henceforth: *Let σ, rather than $\hat{p}$, denote the extended "plus 1" action on $\widehat{\mathbb{N}}$.* Also, let $\mathbb{E}$ denote the universal enveloping semigroup $E(\widehat{\mathbb{N}})$.

Although the the shift acting on $X := \{\mathbf{0}, \mathbf{1}\}^{\mathbb{N}}$ is not isomorphic to $(\sigma\colon \mathbb{E})$, their enveloping semigroups are. The quotient map $\hat{\psi}$ of diagram 17 gives rise to the semigroup-homeomorphism $\Psi\colon \mathbb{E} \twoheadrightarrow E(X)$ of Theorem 14.

19: Full-shift Theorem. *The above Ψ is a semigroup-isomorphism.* ♢

Proof (J.Auslander and N.Markley.) We need to show that Ψ is injective. So, fixing distinct $\alpha,\beta \in \mathbb{E}$, we need to exhibit an $x \in X$ such that $\alpha(x) \neq \beta(x)$. (Of course, "$\alpha(x)$" means $\Psi([\alpha])x$.) Since $\mathbb{E}$ is Hausdorff we can fix disjoint open nbhds $U \ni \alpha$ and $V \ni \beta$. With σ the shift on $\widehat{\mathbb{N}}$, define x by

$$x\lfloor_k = \mathbf{1} \quad \text{iff} \quad \sigma^k \in U$$

for $k \in \mathbb{N}$. Fix nets $\sigma^{k_i} \rightarrowtail \alpha$ and $\sigma^{n_j} \rightarrowtail \beta$. We may assume that all $\sigma^{k_i} \in U$ and all $\sigma^{n_j} \in V$. Chasing definitions yields

$$\begin{aligned} \alpha(x)\lfloor_0 &= \lim_i \sigma^{k_i}(x)\lfloor_0 &= \lim_i x\lfloor_{k_i} &= \mathbf{1} \\ \beta(x)\lfloor_0 &= \lim_j \sigma^{n_j}(x)\lfloor_0 &= \lim_j x\lfloor_{n_j} &= \mathbf{0}\,. \end{aligned}$$

The latter follows by observing that since each σ^{n_j} is in V, it is not in U. ♦

Remark. Actually, for an arbitrary topological group G, let G act on $\{0,1\}^G$ by translation. Then the enveloping semigroup of this action is canonically isomorphic to the Stone-Čech compactification of G. □

The action on $E(T)$

(It is convenient, in this section, to let "orbit" mean non-negative orbit. And to let $E(T)$ be the closure of the non-negative powers of T; thus the identity, I, is a member of $E(T)$.)

Fix a $(T\colon X, x_0)$ in our category and set $\mathbb{E} := E(T)$. Right multiplication $R_T{:}\mathbb{E}{\to}\mathbb{E}$ by $\eta \mapsto \eta T$ is continuous. By definition, the R_T-orbit of the identity, I, is dense in $\mathbb{E}$. Our original triple is a factor of a new triple in our category, via the natural morphism

$$\mathcal{N}\colon (R_T\colon \mathbb{E}, I) \longrightarrow (T\colon X, x_0)\colon \eta \mapsto \eta(x_0)\,.$$

To check that $T\mathcal{N} = \mathcal{N}R_T$, fix η and observe

$$\begin{aligned} T\mathcal{N}(\eta) \stackrel{\text{def}}{=} T\eta(x_0) &= \eta T(x_0) \qquad \text{since } T \in \mathcal{Z}(\mathbb{E}),\\ &= \mathcal{N}(\eta T)\\ &= \mathcal{N}R_T(\eta) \end{aligned}$$

as desired. This natural morphism, when applied with T our universal-lift σ, yields that R_σ is universal –whence *this* curious result:

20: Corollary. *$\widehat{\mathbb{N}}$ is homeomorphic with its enveloping semigroup via the correspondence*

$$E(\widehat{\mathbb{N}}) \ni \eta \longleftrightarrow \eta(0) \in \widehat{\mathbb{N}}\,.$$

In consequence, $\widehat{\mathbb{N}}$ inherits a natural (non-commutative) semigroup operation which extends the "+" operation of the embedded copy of $\mathbb{N}$. ◊

Remark. For $a, b \in \widehat{\mathbb{N}}$, the value of "$ab$" is $\alpha(b)$, where $\alpha \in E(\widehat{\mathbb{N}})$ is the unique member for which $\alpha(0) = a$. □

§C Stone-Čech compactification

This is the general case. For certain special cases, a description using ultrafilters is convenient.

We first describe the evaluation map in the general context. The Stone-Čech compactification of S arises when we specialize the F, below, to be $C[X]$ —the set of continuous functions.

The evaluation map

Given a topological space X let F be a set of functions, all with domain X. That is, for each $f \in F$ we have a topological space Ω_f and map $f\colon X \to \Omega_f$. Define the ***evaluation map***

$$\mathbf{e}\colon X \to \prod_{f\in F} \Omega_f \qquad \mathbf{e}(x) := \langle f \mapsto f(x)\rangle\,.$$

Suppose $\{x_j \mid j \in J\}$ is a net in X and $x_j \rightarrowtail y$. For each continuous $f \in F$, then, $f(x_j) \underset{j}{\rightarrowtail} f(y)$ in Ω_f. Hence $\mathbf{e}(\cdot)$ is continuous at the point $y \in X$ iff each $f \in F$ is continuous at y. Thus

M-a: *The evaluation map $\mathbf{e}(\cdot)$ is continuous IFF each $f \in F$ is continuous*

M-b: *Map $\mathbf{e}(\cdot)$ is injective IFF collection F separates points*

The latter means that $[\forall f\colon\; f(x) = f(z)] \implies x = z$.

When is $\mathbf{e}\colon X\to\mathbf{e}(X)$ a homeomorphism?

Let Π denote the product space $\prod_{f\in F}\Omega_f$. Assume now that F is a separating collection of continuous functions. Here is a condition sufficient to make $\mathbf{e}$ a homeomorphism from X to $\mathbf{e}(X)$. Say that F is a ***Tychonoff family*** if

21: *For each open set $U \subset X$ and point $y \in U$ there exists $g \in F$ such that $g(y) \notin \mathrm{Cl}(g(X \smallsetminus U))$, where the closure is taken in Ω_g.*

To argue that $\mathbf{e}$ is a homeomorphism we need to show it open: Fix an open $U \subset X$ and construct an open set $U' \subset \Pi$ fulfilling

$$\mathbf{e}(U) \quad = \quad \mathbf{e}(X) \cap U'\,,$$

as follows. Given a point $y \in U$, take a function $g \in F$ satisfying (21). For each point $\alpha \in \Pi$ let $\alpha(f)$, an element of Ω_f, denote its "f-th" component. By definition of the product topology, the projection map $P\colon \Pi \to \Omega_g$ of $\alpha \mapsto \alpha(g)$, is continuous. Hence the set

$$\begin{aligned} W^y &:= P^{-1}(\Omega_g \setminus \overline{g(X \setminus U)}) \\ &= \{\alpha \in \Pi \mid \alpha(g) \in \Omega_g \setminus \overline{g(X \setminus U)}\} \end{aligned}$$

is an open subset of Π. Note that $\mathbf{e}(x)$ is in W^y if and only if $g(x) \in \Omega_g \setminus \overline{g(X \setminus U)}$. Hence

$$\mathbf{e}(y) \in W^y \qquad \text{and} \qquad \mathbf{e}(X \setminus U) \cap W^y = \varnothing.$$

Thus the open set $U' := \bigcup_{y \in U} W^y$ is disjoint from $\mathbf{e}(X \setminus U)$, as desired. This shows

M-c: *If F is a Tychonoff family then $\mathbf{e}$ is a homeomorphism of X onto $\mathbf{e}(X)$.*

Constructing the compactification

A ***compactification*** of X is a pair $(\mathbf{e} : K)$ where K is a compact Hausdorff space and $\mathbf{e}\colon X \to \mathbf{e}(X) \subset K$ is a *homeomorphism* onto a dense subset of K.

Let I be the topologized unit interval and let $C[X]$ denote the collection of continuous functions from $X \to I$. Say that X is a ***Tychonoff space*** if $C[X]$ is a Tychonoff family and (this is non-traditional usage) points are closed. Together, these imply that X is Hausdorff, indeed completely-regular ($T_{3.5}$).

The partial order on compactifications

Given two compactifications $e\colon X \hookrightarrow K$ and $f\colon X \hookrightarrow L$, say that

$$(f : L) \ \geq \ (e : K)$$

if there exists a continuous map $\varphi\colon L \to K$ such that $\varphi \circ f = e$. Of necessity, such a φ is unique since $f(X)$ is dense in L. (Since K is Hausdorff, nets have unique limits, etc.) Also, $e(X)$ is dense in K and so $\varphi(L)$ is a dense compact subset of K; thus φ is surjective.

Relation $\geq$ is transitive. Moreover, if

$$(f : L) \overset{\varphi}{\geq} (e : K) \quad \text{and} \quad (e : K) \overset{\psi}{\geq} (f : L)$$

then $(f : L) \geq (f : L)$ via $\psi \circ \varphi$. Thus $\psi \circ \varphi = Id_L$, by uniqueness. Similarly, $\varphi \circ \psi = Id_K$. Hence φ and ψ are homeomorphisms carrying f to e and vice versa. This is a reasonable definition of ***isomorphism*** between two compactifications.

22: Theorem. *Every Tychonoff space X has a compactification $X \overset{\mathbf{e}}{\hookrightarrow} \hat{X}$ with the following "compact space lifting property".*

23: *For each compact Hausdorff space K and continuous map $\varphi\colon X \to K$ there is a continuous "lift" $\hat{\varphi}(\hat{X})K$ satisfying $\hat{\varphi} \circ \mathbf{e} = \varphi$. (This $\hat{\varphi}$ is unique since $\mathbf{e}(X)$ is, by hypothesis, dense in $\hat{X}$.)*

Moreover, if X is compact then X and $\hat{X}$ are homeomorphic via $\mathbf{e}$. ◊

24: Corollary. *The Stone-Čech compactification is $\geq$ every other compactification. In particular, it is unique (upto isomorphism).* ◊

Proof. Suppose $e\colon X \hookrightarrow K$ and $f\colon X \hookrightarrow L$ both have lifting property, (23). Then there exist continuous maps

$$\begin{array}{ccc} K & \xrightarrow{\hat{f}} & L \\ {\scriptstyle e}\uparrow & & \| \\ X & \xrightarrow{f} & L \end{array} \qquad \text{and} \qquad \begin{array}{ccc} L & \xrightarrow{\hat{e}} & K \\ {\scriptstyle f}\uparrow & & \| \\ X & \xrightarrow{e} & K \end{array}$$

Thus $\hat{e}(\hat{f}e) = \hat{e}f = e$. Hence $\hat{e}\hat{f}\colon K\circlearrowleft$ is the identity on $e(X)$, a dense subset of X. Since K is Hausdorff, the continuity of $\hat{e}\hat{f}$ forces it to be the identity map on all of K.

Similarly, $\hat{f}\hat{e}\colon L\circlearrowleft$ is the identity map. Thus $\hat{e}$ and $\hat{f}$ are homeomorphisms (using compactness and Hausdorff again) carrying f to e and vice versa. ♦

There is much more of this to be typed up. The notes are in ERGODIC NB NB: *Topological Dynamics* and are handwritten.

NOTE: The embedding of a space into its Stone-Čech compactification is essentially the same as the embedding of a vector space into its double-dual.

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